HOMOTOPY DECOMPOSITIONS AND K–THEORY OF BOTT TOWERS

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Abstract. We describe Bott towers as sequences of toric manifolds M^k, and identify the omniorientations which correspond to their original construction as toric varieties. We show that the suspension of M^k is homotopy equivalent to a wedge of Thom complexes, and display its complex K-theory as an algebra over the coefficient ring. We extend the results to KO-theory for several families of examples, and compute the effects of the realification homomorphism; these calculations breathe geometric life into Bahri and Bendersky's analysis of the Adams Spectral Sequence [2]. By way of application we investigate stably complex structures on M^k, identifying those which arise from omniorientations and those which are almost complex. We conclude with observations on the rôle of Bott towers in complex cobordism theory.

1. Introduction

In their 1950s study of loops on symmetric spaces, Bott and Samelson [4] introduced a remarkably rich and versatile family of smooth manifolds. Various special cases were treated in different contexts during the following three decades, until Grossberg and Karshon [13] offered a description as complex algebraic varieties in 1994. They referred to their constructions as *Bott towers*, and addressed issues of representation theory and symplectic geometry. Our purpose here is to offer the alternative viewpoint of algebraic topology. We consider Bott towers $(M^k : k \leq n)$ of height n, and discuss homotopy decompositions of the suspensions ΣM^k; these provides further evidence that the spaces of complex geometry are often stably homotopy equivalent to wedges of Thom complexes, as we have argued elsewhere [12]. We investigate the real and complex K-theory of the M^k, casting geometric light on recent calculations of Bahri and Bendersky [2] which were originally conducted in the algebraic underworld of the Adams Spectral Sequence.

Given a commutative ring spectrum E, we denote the reduced and unreduced cohomology algebras of any space X by $E^*(X)$ and $E^*(X_+)$ respectively. So $E^*(S^n)$ is a free module over the coefficient ring E_* on a single n-dimensional generator s_n^E, defined by the unit of E. In particular, we use this notation for the integral Eilenberg-Mac Lane spectrum H and the complex K-theory spectrum K. Real K-theory requires the most detailed calculations, so we abbreviate s_n^{KO} to s_n whenever possible. We require multiplicative maps $f\colon E \to F$ of ring spectra to preserve the units, so that $f(s_n^E) = s_n^F$ for all n; complexification

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$c\colon KO \to K$ is an important example. We adopt similar conventions for Thom classes t^E, which also play a major rôle. Given an E-orientable n–dimensional vector bundle γ, we insist that t^E should lie in $E^n(T(\gamma))$, and restrict to s_n^E on the fibre. Alternative choices of dimension are, of course, available for periodic spectra such as K and KO, but we believe that our chosen convention leads to the least confusion.

With the single exception of KO, the spectra we use are complex oriented by an appropriate choice of first Chern class v^E in $E^2(CP^\infty)$; by definition, v^E restricts to s_2^E on CP^1. We also insist that E_* be concentrated in even degrees.

The contents of our sections are as follows.

In Section 2 we establish our notation, and recall well-known computations for the E-cohomology of certain sphere bundles Y over complexes with cells in even dimensions. We record a homotopy decomposition of ΣY, and apply the results to K-theory and integral cohomology. We introduce Bott towers as iterated sphere bundles in Section 3, and apply the previous section to describe their E-cohomology algebras, and splittings of their suspensions. We also consider their stable tangent bundles, and introduce a cofiber sequence relating pairs of towers. Bott towers masquerade as toric varieties, and we discuss their associated properties in Section 4; we adapt the viewpoint of Grossberg and Karshon, and pay particular attention to the corresponding complex structures. Our calculations with KO-theory begin in Section 5, where we focus on dimensions 2 and 4. We obtain complete descriptions of the KO_*-algebra structure in all cases. These results provide a springboard for our most comprehensive calculations, which occupy Section 6; we consider all dimensions, but specialise to two particular families of cases. Again, we obtain complete information about KO_*-algebra structures, but find that certain products are particularly complicated to describe explicitly. We relate our results to the pioneering work of Bahri and Bendersky. Finally, in Section 7, we apply these calculations to the enumeration of a collection of stably complex structures, which arise from our study of Bott towers as toric manifolds. Such structures are of key importance to understanding their rôle in complex cobordism theory.

The idea of studying Bott towers in this context first emerged during discussions with Victor Buchstaber, made possible by Aeroflot's abandonment of flights out of Manchester in 1996. The second author announced most of the results at the Conference on Algebraic Topology in Gdansk, Poland, during June 2001, where Taras Panov and his colleagues offered many helpful suggestions as we strolled the Baltic beaches. We apologise to them all for our protracted attempts to produce a final document, and give thanks to Adrian Dobson for identifying several errors in various intermediate versions.

2. 2–Generated Complexes

It is convenient to work with connected CW-complexes X whose integral cohomology ring $H^*(X;\mathbb{Z})$ is generated by a linearly independent set of 2–dimensional elements $x_1, \ldots, x_m$. We describe such an X as being *2–generated*, and note that $H^2(X;\mathbb{Z})$ is isomorphic to the integral lattice $\mathbb{Z}^m$; we refer to the elements x_j as the *2–generators* of X, and to m as its *2–rank*. We follow

combinatorial convention by abbreviating the set $\{1,\ldots,m\}$ to $[m]$, and denote the product $\prod_R x_j$ by x_R for any subset $R\subseteq[m]$. The first Chern class v^H defines a canonical isomorphism between the multiplicative group of complex line bundles over X and $H^2(X;\mathbb{Z})$, and so determines line bundles γ_j such that $v^H(\gamma_j)=x_j$, for $1\leq j\leq m$. In general, it assigns the m-tuple $(a(1),\ldots,a(m))$ to the tensor product

$$\gamma_1^{a(1)}\otimes\cdots\otimes\gamma_m^{a(m)}. \tag{2.1}$$

By definition, X lies in the category of CW-complexes whose cells are even dimensional. Various observations of Hoggar [14] therefore apply to the abelian group structure of $KO^*(X)$, and are relevant to parts of Sections 5 and 6.

Given any of our complex oriented ring spectra E, the Chern classes $v^E(\gamma_j)=v_j^E$ lie in $E^2(X)$ for all $1\leq j\leq m$. The corresponding Atiyah-Hirzebruch spectral sequence collapses for dimensional reasons, and identifies $E^*(X)$ as a free E_*-module, spanned by the monomials v_R^E; in other words, it is generated by $v_1^E,\ldots,v_m^E$ as an E_*-algebra. An important, if atypical, example is provided by CP^n. Then $v=v^E(\zeta(n))$ is the first Chern class of the Hopf line bundle $\zeta(n)$, and the canonical isomorphism

$$E^*(CP^n_+)\;\cong\;E_*[[v]]/(v^{n+1}) \tag{2.2}$$

confirms that CP^n has the single 2–generator v. In order to emphasise that we are working over CP^n, we sometimes denote v by $v(n)$; thus $v(1)$ and s_2^E are interchangeable. In the cases $E=H$ and K, we write v as x and u respectively.

The following results are well-known, and are usually obtained by applying standard methods of Borel and Hirzebruch [3]. Our immediate interests, however, are homotopy theoretic, and involve the stable triviality of certain cofibre sequences of 2–generated complexes and associated Thom spaces. We therefore take the opportunity to establish our notation by outlining proofs in this alternative language.

We assume that X is 2–generated, and write γ for the line bundle (2.1). We let Y denote the total space $S(\mathbb{R}\oplus\gamma)$ of the 2–sphere bundle obtained from γ by the addition of a trivial real line bundle, and write p for the projection onto X. Whenever X is a smooth manifold, we may assume that Y is also.

Lemma 2.3. *The E_*-algebra $E^*(Y_+)$ is a free module over $E^*(X_+)$ on generators* 1 *and v_{m+1}^E, which have dimensions* 0 *and* 2 *respectively; the multiplicative structure is determined by the single relation*

$$(v_{m+1}^E)^2=v^E(\gamma)v_{m+1}^E, \tag{2.4}$$

and v_{m+1}^E restricts to s_2^E on the fibre $S^2\subset Y$.

Proof. The sphere bundle $S(\mathbb{R}\oplus\gamma)$ admits a section r, given by $+1$ in the summand $\mathbb{R}$, and the quotient of the total space by the image of r is canonically homeomorphic to the Thom complex $T(\gamma)$ [20]. In the resulting cofibre sequence

$$X\xrightarrow{r}Y\xrightarrow{q}T(\gamma), \tag{2.5}$$

the quotient map q identifies the fibres $S^2 \subset Y$ and $S^2 \subset T(\gamma)$, and r has left inverse p. The standard coaction of X on $T(\gamma)$ interacts with the diagonal on Y by the commutative square

$$\begin{array}{ccc} Y & \xrightarrow[q]{} & T(\gamma) \\ {\scriptstyle\delta}\big\downarrow & & \big\downarrow{\scriptstyle\delta} \\ Y \times Y & \xrightarrow{(p,q)} & X_+ \wedge T(\gamma) \end{array} \quad . \tag{2.6}$$

The E-cohomology sequence induced by (2.5) is split by p^*, and is therefore short exact. The Chern class v^E induces a canonical Thom class $t^E \in E^2(T(\gamma))$, and so determines a Thom isomorphism $E^{*-2}(X_+) \cong E^*(T(\gamma))$, which identifies $E^*(Y_+)$ as the free $E^*(X_+)$–module on generators 1 and $v^E_{m+1} = q^*t^E$. The diagram (2.6) confirms that products of the form $p^*(x)v^E_{m+1}$ may be written as $q^*(xt^E)$ for any $x \in E^*(X)$; so the action of v^E_{m+1} is by multiplication in $E^*(Y)$. Since $\delta^*(v^E(\gamma) \otimes t^E) = (t^E)^2$, the formula for $(v^E_{m+1})^2$ follows. □

An obvious consequence of Lemma 2.3 is that Y is also 2–generated, and has 2–rank $m+1$. The Chern class $v^E(\gamma)$ may be expanded in terms of the E_*-basis $v^E_1, \ldots, v^E_m$, using the associated formal group law F^E and its n-series $[n]^E$. We obtain

$$v^E(\gamma) = F^E([a(1)]^E, \ldots, [a(m)]^E) \tag{2.7}$$

in $E^2(x)$.

The universal example of Lemma 2.3 is given by $X = CP^\infty$ and $\gamma = \zeta$; it follows that $T(\gamma)$ is also homeomorphic to CP^∞, and that Y is homotopy equivalent to $CP^\infty \vee CP^\infty$. Then $E^*(Y_+)$ is free over $E_*[[v]]$ on generators 1 and v', with $(v')^2 = vv'$. The general case may be deduced from this example by pulling back along the classifying map for γ. Of course, we may restrict the universal example to any skeleton $X = CP^n$, in which case $T(\gamma)$ is CP^{n+1}.

There is a second section $\widetilde{r}\colon X \to Y$, defined by $-1 \in \mathbb{R}$. The resulting composition $q \cdot \widetilde{r}\colon X \to T(\gamma)$ reduces to the inclusion of the zero-section, giving $\widetilde{r}^* t^E = v^E(\gamma)$.

The usual approach to Lemma 2.3 proceeds by identifying $S(\mathbb{R} \oplus \gamma)$ with its *projective form* $CP(\mathbb{C} \oplus \gamma)$. The corresponding canonical line bundle has first Chern class v^E_{m+1}, and is isomorphic to γ_{m+1}; it restricts to the Hopf bundle $\zeta(1)$ over the fibre CP^1. So γ_{m+1} is a summand of the pullback $\mathbb{C} \oplus \gamma$ over Y, and has orthogonal complement $\overline{\gamma}_{m+1} \otimes \gamma$ with respect to the standard inner product. The associated splitting

$$\mathbb{C} \oplus p^*\gamma \;\cong\; \gamma_{m+1} \oplus \big(\overline{\gamma}_{m+1} \otimes p^*\gamma\big) \tag{2.8}$$

gives rise to the relation (2.4), and will be useful in Section 7.

The cofibre sequence (2.5) also leads to the familiar relationship between the homotopy types of X and Y.

Proposition 2.9. *There is a homotopy equivalence*

$$h\colon \varSigma Y \longrightarrow \varSigma X \vee \varSigma T(\gamma)$$

of suspensions.

Proof. We define h as the sum $\Sigma p + \Sigma q$, and construct a homotopy inverse $\Sigma X \vee \Sigma T(\gamma) \to \Sigma Y$ by forming the wedge of Σr with the map $l\colon \Sigma T(\gamma) \to \Sigma Y$ which collapses the standard copy of X in $T(\mathbb{R} \oplus \gamma)$. □

The equivalence h induces an isomorphism in E-cohomology, which realises the module structures of Lemma 2.3 by splitting $E^*(Y_+)$ as $E^*(X_+) \oplus (v^E_{m+1})$. In the universal example, h is a self equivalence of $\Sigma CP^\infty \vee \Sigma CP^\infty$ and desuspends.

We shall need an extension of Lemma 2.3, in the situation when X itself is the total space of a bundle θ over S^2, with fibre X'. We write γ' for the pullback of γ to X', and Y' for the total space $S(\mathbb{R} \oplus \gamma')$; thus Y' is also the fibre of the projection $Y \to S^2$.

Proposition 2.10. *With the data above, there is a homotopy commutative ladder of cofibre sequences*

$$\begin{array}{ccccc} T(\gamma') & \xrightarrow{i} & T(\gamma) & \xrightarrow{f} & \Sigma^2 T(\gamma') \\ {\scriptstyle q'}\uparrow & & {\scriptstyle q}\uparrow & & \uparrow{\scriptstyle \Sigma^2 q'} \\ Y' & \xrightarrow[i]{} & Y & \xrightarrow[f]{} & \Sigma^2 Y'_+ \end{array} , \tag{2.11}$$

where the maps i are induced by inclusion of the fibre, and the maps f are quotients.

Proof. We may construct X from two copies of $\mathbb{D}^2 \times X'$ by identifying them along their boundaries $S^1 \times X'$ via the characteristic function of θ. Then $X/i(X')$ is homeomorphic to $\Sigma^2 X'_+$. The same argument applies to $Y/i(Y')$, yielding cofibre sequences

$$X' \xrightarrow{i} X \xrightarrow{f} \Sigma^2 X'_+ \qquad \text{and} \qquad Y' \xrightarrow{i} Y \xrightarrow{f} \Sigma^2 Y'_+ . \tag{2.12}$$

The sections $r'\colon X' \to Y'$ and $r\colon X \to Y$ are compatible with the inclusions i, and the ladder follows by taking quotient maps q' and q. □

The naturality of the ladder (2.11) leads to a commutative square

$$\begin{array}{ccc} T(\gamma) & \xrightarrow[f]{} & \Sigma^2 T(\gamma') \\ {\scriptstyle \delta}\downarrow & & \downarrow{\scriptstyle \epsilon} \\ X \wedge T(\gamma) & \xrightarrow{f \wedge 1} & \Sigma^2 (X'_+) \wedge T(\gamma) \end{array} , \tag{2.13}$$

where ϵ is the Thom complexification of the bundle map obtained by pulling $\mathbb{R}^2 \times \gamma$ back along the restricted diagonal $X' \to X' \times X$. Alternatively, the square may be considered as the quotient of the reduced diagonal $T(\gamma) \to X \wedge T(\gamma)$ by its restriction $T(\gamma') \to X' \wedge T(\gamma)$.

The first sequence of (2.12) induces the Wang long exact sequence of θ in E-cohomology, for any multiplicative spectrum E. Standard homotopy theoretic arguments [23] show that the connecting map $\Sigma^2 X'_+ \to \Sigma X'$ is induced from the characteristic map $S^1 \times X' \to X'$ by suspension.

We shall apply these facts in the particular cases $E = H$ and K, denoting the elements v^E_{m+1} by x_{m+1} and g_{m+1} respectively. We write the coefficients of

complex K-theory as the ring of Laurent series

$$K_* := \mathbb{Z}[z, z^{-1}],$$

where z lies in K_2 and is represented by the virtual Hopf line bundle over S^2. So zg_j is represented by the virtual bundle $\gamma_j - \mathbb{C}$ in $K^0(X)$, for $1 \leq j \leq m$. Complex conjugation acts on K_* by $\overline{z} = -z$, and on the algebra generators by

$$\overline{g}_j \;=\; \overline{\gamma}_j g_j \;=\; g_j/(1+zg_j) \;=\; \sum_{i=0}^{\infty}(-z)^i g_j^{i+1}; \tag{2.14}$$

the Chern character embeds $K^*(X)$ in the ring $H^*(X;\mathbb{Q}[z,z^{-1}])$ by $ch(g_j) = z^{-1}(e^{zx_j}-1)$, for $1 \leq j \leq m$.

The cases H and K correspond to the additive and multiplicative formal group laws respectively. The Chern classes (2.7) are given by

$$\begin{aligned} v^H(\gamma) &= a(1)x_1 + \cdots + a(m)x_m \quad \text{and} \\ v^K(\gamma) &= z^{-1}\Big(\prod_{j\leq m}(1+zg_j)^{a(j)} - 1\Big), \end{aligned} \tag{2.15}$$

and are compatible under the action of the Chern character.

3. Bott Towers

In this section we consider the algebraic topology of Bott towers, extending our work [18] on bounded flag manifolds; our methods complement the more geometric approach of [9]. We give an inductive construction as a family of 2–generated smooth oriented manifolds M^k, and describe their cohomology rings for any of our complex oriented ring spectra E. We obtain an elementary decomposition of their suspensions ΣM^k into a wedge of Thom complexes, and consider two natural complex structures on their stable tangent bundles.

Given any integer $k \geq 1$, we assume that a $(k-1)$*th stage* M^{k-1} has been constructed as a smooth oriented $2(k-1)$–dimensional manifold with 2–generators v_j^E, and line bundles γ_j such that $v^E(\gamma_j) = v_j^E$, for $1 \leq j \leq k-1$. Using the notation of (2.1), we write $\gamma(a_{k-1})$ for the complex line bundle

$$\gamma_1^{a(1,k)} \otimes \cdots \otimes \gamma_{k-1}^{a(k-1,k)}$$

associated to the $(k-1)$-tuple $a_{k-1} = (a(1,k), \ldots, a(k-1,k))$ in $\mathbb{Z}^{k-1}$. Fixing a_{k-1}, we refer to $\gamma(a_{k-1})$ as the k*th bundle* of the construction, and define M^k to be the total space of the smooth 2–sphere bundle of $\mathbb{R} \oplus \gamma(a_{k-1})$, oriented by the outward pointing normal and the complex structure on $\gamma(a_{k-1})$. By Lemma 2.3, we deduce that M^k has 2–generators v_j^E for $1 \leq j \leq k$, where v_k^E is the pullback of the Thom class $t_k^E \in E^2(T(\gamma(a_{k-1})))$ along the collapse map q_k. Moreover, t_k^E is the first Chern class of a canonical line bundle λ_{k-1} over $T(\gamma(a_{k-1}))$, so $v_k^E = v^E(\gamma_k)$, where γ_k is defined as $q_k^*\lambda_{k-1}$. Henceforth, we abbreviate $T(\gamma(a_j))$ to $T(a_j)$ for each $1 \leq j \leq n$.

In order to get off the ground, it is convenient to write the one-point space as M^0, so that the first bundle is trivial and $x_0 = 0$. Then M^1 is a 2–sphere, compatibly oriented with the complex structure on CP^1, and γ_1 is the Hopf line bundle $\zeta(1)$. The cohomology ring $E^*(S^2_+)$ is isomorphic to $E_*[v]/(v^2)$, where

$v = v^E(\zeta(1)) = s_2^E$, and S^2 is 2–generated with 2–rank 1. Of course the second bundle $\gamma(a_1)$ is isomorphic to $\zeta(1)^{a(1,2)}$ for some 1-term sequence $a_1 = (a(1,2))$.

The construction is now complete, and the kth stage depends only on the integral sequences $(a_1, \dots, a_{k-1})$, which contain $k(k-1)/2$ integers $a(i,j)$, for $1 \leq i < j \leq k-1$. It is occasionally helpful to interpret a_0 as empty, and to write the first bundle $\mathbb{C}$ as $\gamma(a_0)$.

We refer to the sequence $(M^k : k \leq n)$ of oriented manifolds as a *Bott tower* of *height* n (which may be infinite); it is determined by the *list* $a = (a_1, \dots, a_{n-1})$ of $n(n-1)/2$ integers. If we choose the projective form of M^k at every stage, we obtain a tower of nonsingular algebraic varieties, whose orientations coincide with those decribed above. Every Bott tower involves projections $p_k \colon M^k \to M^{k-1}$, sections r_k and $\widetilde{r}_k \colon M^{k-1} \to M^k$, and quotient maps $q_k \colon M^k \to T(a_{k-1})$, for each $1 \leq k \leq n$.

The cohomological structure of M^k is given as follows.

Proposition 3.1. *For any complex oriented ring spectrum E, the E_*-algebra $E^*(M^k_+)$ is isomorphic to $E_*[v_1^E, \dots, v_k^E]/I_k^E$, where I_k^E denotes the ideal*

$$\big((v_j^E)^2 - v^E(\gamma)v_j^E : 1 \leq j \leq k\big);$$

in particular, $E^{2r}(M^k_+)$ is the free E_*-module generated by the monomials v_R^E, as $R \subseteq [k]$ ranges over the subsets of cardinality r, and $E^*(M^k_+)$ has total rank 2^r.

Proof. The multiplicative structure follows from $k-1$ applications of Lemma 2.3; the resulting relations imply the additive structure immediately. □

In the cases $E = H$ and K, we denote the elements v_j^E by x_j in $H^2(M^k_+;\mathbb{Z})$ and g_j in $K^2(M^k_+)$ respectively, for $1 \leq j \leq k$. The ideals I_k^H and I_k^K are then described explicitly by (2.15). The structure of $H^*(M^k_+;\mathbb{Z})$ shows that the Euler characteristic of M^k is 2^k, and is independent of a; this may also be confirmed by straightforward geometric argument.

By way of example we consider the tower $(B_k : 0 \leq k)$, whose list satisfies $a_k = (0, \dots, 0, 1)$ for all $k \geq 1$. We studied this example in [18], where we explained its significance for complex cobordism theory. In later work [5] we interpreted the points of B_k as complete flags $0 < U_1 < \cdots < U_n < \mathbb{C}^{k+1}$, bounded below by the standard flag in the sense that the first j standard basis vectors lie in U_{j+1}, for each $1 \leq j \leq k$. The resulting description of B_k as a *bounded flag manifold* corresponds to the projective form $CP(\mathbb{C} \oplus \gamma_{k-1})$, and displays B_k as a toric variety.

We may now describe our homotopy theoretic decomposition of ΣM^k.

Proposition 3.2. *Given any Bott tower $(M^k : k \leq n)$, there is a homotopy equivalence*

$$h_k \colon \Sigma M^k \longrightarrow \Sigma S^2 \vee \Sigma T(a_1) \vee \cdots \vee \Sigma T(a_{k-1}),$$

for each $1 \leq k \leq n$.

Proof. It suffices to apply Proposition 2.9 $k-1$ times; S^2 appears as $T(a_0)$. □

With respect to Proposition 3.1, the homotopy equivalence h_k induces the additive splitting

$$E^*(M^k) \;\cong\; \langle v^E_{\leq 1}\rangle \oplus \cdots \oplus \langle v^E_{\leq k}\rangle$$

where $\langle v^E_{\leq j}\rangle$ denotes the free E_*-submodule generated by those monomials v^E_R for which $R \subseteq [j]$ and $j \in R$. By construction, $\langle v^E_{\leq j}\rangle$ is the image of $E^*(T(a_{j-1}))$ under the injection $p^*_k \cdots p^*_{j+1} q^*_j$, for each $1 \leq j \leq k$; it is split by $l^*_{j-1} r^*_j \cdots r^*_k$, where l^*_{j-1} is induced by the map $\Sigma T(a_{j-1}) \to \Sigma M^{j-1}$ which collapses the standard copy of M^{j-1} in $T(\mathbb{R} \oplus \gamma(a_{j-1}))$.

It is worth commenting on aspects of the case $k = 2$, which is influenced by the fact that the isomorphism class of the $SO(3)$-bundle $\mathbb{R} \oplus \zeta(1)^{a(1,2)}$ depends only on the parity of $a(1,2)$. So there are diffeomorphisms $M^2 \to S^2 \times S^2$ when $a(1,2) = 2b$ is even, and $M^2 \to S(\mathbb{R} \oplus \zeta(1))$ when $a(1,2) = 2b+1$ is odd. In E-cohomology, they induce isomorphisms

$$\begin{aligned} E_*[v_1, v_2]/\left(v_1^2, v_2^2 - 2bv_1v_2\right) \;&\cong\; E_*[w_1, w_2]/(w_1^2, w_2^2) \quad \text{and} \\ E_*[v_1, v_2]/\left(v_1^2, v_2^2 - (2b+1)v_1v_2\right) \;&\cong\; E_*[w_1, w_2]/\left(w_1^2, w_2^2 - w_1w_2\right), \end{aligned} \tag{3.3}$$

(omitting the superscripts E), which are determined by the 2×2 matrices of their actions on the column vector (v_1, v_2). Such matrices are exemplified by $\left(\begin{smallmatrix} 1 & 0 \\ b & 1 \end{smallmatrix}\right)$, for any integer b.

We shall be particularly interested in the stable tangent bundle of M^k in Section 7 below. As explained by Szczarba [21], there is an explicit isomorphism

$$\tau(M^k) \oplus \mathbb{R} \;\cong\; \mathbb{R} \oplus \bigoplus_{j=1}^{k} \gamma(a_{j-1}) \tag{3.4}$$

of $SO(2k+1)$–bundles, which determines a stably almost complex structure τ' on M^k. Since (3.4) extends over the 3–disk bundle of $\mathbb{R} \oplus \gamma(a_{k-1})$, this structure bounds. On the other hand, the projective form of M^k is a nonsingular complex algebraic variety, whose tangent bundle admits the canonical complex structure described in Section 4. The fact that its stabilisation differs from (3.4) is one of our motivations for Section 7.

Given a Bott tower of height n, we turn our attention to the projection $p_{n,k}\colon M^n \to M^k$, defined as the composition $p_{k+1} \cdots p_n$ for some $k \geq 1$. This is also a smooth bundle, whose fibre we wish to identify.

Proposition 3.5. *The fibre of $p_{n,k}$ is the $(n-k)$th stage of a Bott tower $((M')^j : j \leq n-k)$; it is determined by the list $(a'_1, \ldots, a'_{n-k-1})$, where a'_j is formed from a_{j+k} by deleting the first k entries, for each $1 \leq j \leq n-k-1$.*

Proof. When we restrict the bundle $\gamma(a_k)$ to a point $(M')^0$ in M^k, we obtain the trivial bundle $\mathbb{C}$, and M^{k+1} pulls back to the fibre S^2 of p_{k+1}; we label this fibre $(M')^1$. We repeat the pullback procedure over $(M')^1$, and continue until we reach M^{n-1}. We find that γ_j restricts trivially to $(M')^{n-k-1}$ for $1 \leq j \leq k$, and to γ'_{j-k} for $k < j \leq n-1$. Thus $\gamma(a_{n-1})$ restricts to $\gamma'(a'_{n-k-1})$, where $a'_{n-k-1} = (a(k+1,n), \ldots, a(n-1,n))$, and M^n pulls back to $S(\mathbb{R} \oplus \gamma'(a'_{n-k-1}))$,

which we label $(M')^{n-k}$. The construction ensures that $(M')^{n-k}$ is the inverse image of $(M')^0$ under $p_{n,k}$, and is therefore the required fibre. $\square$

Corollary 3.6. *For each $1<k<n$, there is a commutative ladder of cofibre sequences*

$$\begin{tikzcd} T(a'_{k-2}) \arrow[r, "i"] & T(a_{k-1}) \arrow[r, "f"] & \Sigma^2 T(a'_{k-2}) \\ (M')^{k-1} \arrow[u, "q'_{k-1}"] \arrow[r, "i"'] & M^k \arrow[u, "q_k"] \arrow[r, "f"'] & \Sigma^2(M')^{k-1}_+ \arrow[u, "\Sigma^2 q'_{k-1}"'] \end{tikzcd} \tag{3.7}$$

In E-cohomology, the homomorphisms induced by the upper sequence satisfy $i^*t_k^E=(t')^E_{k-1}$, and $f^*(\Sigma^2 i^* w(t')^E_{k-1})=v_1^E w t_k^E$ for every $w\in E^*(M^{k-1})$. In the lower sequence they satisfy $i^*v_j^E=(v')^E_{j-1}$ for each $2\le j\le k$, with $i^*v_1^E=0$, and $f^*(\Sigma^2 i^* v_R^E)=v_1^E v_R^E$ for every $R\subseteq\{2,\dots,k\}$, with $f^*s_2^E=v_1^E$.

Proof. The ladder arises by combining Proposition 3.5 with Proposition 2.10, where X is M^{k-1} and Y is M^k. Since the upper i arises from a bundle map it satisfies $i^*t_k^E=(t')^E_{k-1}$, yielding $i^*v_k^E=(v')^E_{k-1}$; the corresponding result holds for $j<k$ by projection onto M^j, noting that $(t')^E_0=0$. Pulling $s_2^E\otimes wt_k^E$ back around (2.13) confirms that $f^*(\Sigma^2 i^* w(t')^E_{k-1})=v_1^E w t_k^E$ in $E^*(T(a_{k-1})$, and applying (2.6) leads to the formula for f^* on $E^*(\Sigma^2(M')^{k-1}_+)$. $\square$

Since all the spaces on view in Corollary 3.6 are 2-generated, the horizontal cofibre sequences are cohomologically split. The formulae for i^* and f^* show that the splitting of $E^*(M^k_+)$ take the form

$$\begin{aligned} &E_*[v_1^E,\dots,v_k^E]/I_k^E \cong \\ &\quad E_*[(v')^E_1,\dots,(v')^E_{k-1}]/(I')^E_{k-1}\oplus v_1^E E_*[(v')^E_1,\dots,(v')^E_{k-1}]/(I')^E_{k-1}, \end{aligned} \tag{3.8}$$

and subsumes the splitting of $E^*(T(a_{k-1}))$ as

$$\langle v^E_{\le k}\rangle \cong \langle (v')^E_{\le k-1}\rangle \oplus v_1^E\langle (v')^E_{\le k-1}\rangle.$$

4. Toric Structures

We now describe the stages of a Bott tower $(M^k : k\le n)$ as toric manifolds, in the sense of Davis and Januszkiewicz; we continue to assume that the tower is determined by the list $a=(a_1,\dots,a_{n-1})$. We use the language of [6] to record the salient properties, and discuss the relationship with Grossberg and Karshon's construction [13] of the M^k as complex manifolds.

We write the k-dimensional torus as T^k and denote a generic point t by $(t_1,\dots,t_k)$, where t_i lies in the unit circle $T\subset\mathbb{C}$ for each $1\le i\le k$. So T^k is naturally embedded in $\mathbb{C}^k$, on which it acts coordinatewise, by multiplication; this is the *standard action*, whose quotient is the nonnegative orthant $\mathbb{R}^k_{\geqslant}$. We study the standard action of T^{2k} on $(S^3)^k$, induced by embedding the latter in $\mathbb{C}^{2k}$ as the subspace

$$\{(y_1,z_1,\dots,y_k,z_k) : y_i\overline{y}_i+z_i\overline{z}_i=1 \text{ for } 1\le i\le k\}. \tag{4.1}$$

When $k=1$ the quotient of this action is a curvilinear 1–simplex, or interval, $I=\{(r,s): r^2+s^2=1\}$ in $\mathbb{R}^2_{\geqslant}$, so for general k it is a curvilinear cube $I^k\subset\mathbb{R}^{2k}_{\geqslant}$.

Given a, we define the k–dimensional subtorus $T^k(a)<T^{2k}$ to consist of elements

$$\begin{aligned}\{(u_1,u_1,u_2,u_1^{-a(1,2)}u_2,\dots,u_k,u_1^{-a(1,k)}\dots u_{k-1}^{-a(k-1,k)}u_k):\\ u_i\in T\ \text{ for }\ 1\le i\le k\},\end{aligned} \tag{4.2}$$

for each $k\le n$. So $T^k(a)$ acts freely on $(S^3)^k$, and the quotient space Q_k is a smooth $2k$–dimensional manifold. Moreover, the k–torus $T^{2k}/T^k(a)$ acts on Q_k, and has quotient I^k; with respect to this action, Q_k is a toric manifold. We abbreviate $T^{2k}/T^k(a)$ to T^k_a whenever it acts on Q_k in this fashion.

Proposition 4.3. *Given any $1\le k\le n$, there is an orientation preserving diffeomorphism $\phi_k\colon Q_k\to M^k$; it pulls γ_j back to the line bundle*

$$(S^3)^k\times_{T^k(a)}\mathbb{C}\longrightarrow Q_k$$

for each $1\le j\le k$, where $T^k(a)$ acts on $\mathbb{C}$ by $w\mapsto u_j^{-1}w$.

Proof. We proceed by induction on k, noting that ϕ_1 is defined by factoring out the action of $T^1(a)=T$ on the domain of the canonical projection $S^3\to CP^1$. By definition, the line bundle γ_1 pulls back to

$$S^3\times_T\mathbb{C}\longrightarrow Q_1,$$

where T acts on $\mathbb{C}$ by $w\mapsto u_1^{-1}w$.

For any $k\ge 1$ we assume that ϕ_k has been constructed with the stated properties. So $\phi_k^*\gamma(a_k)$ is given by

$$(S^3)^k\times_{T^k(a)}\mathbb{C}\longrightarrow Q_k,$$

where $T^k(a)$ acts on $\mathbb{C}$ by $w\mapsto u_1^{-a(1,k+1)}\dots u_k^{-a(k,k+1)}w$. It follows that the projectivisation $CP(\phi_k^*(\mathbb{C}\oplus\gamma(a_k)))$ coincides with Q_{k+1}, and we define ϕ_{k+1} to be the resultant bundle map to $CP(\mathbb{C}\oplus\gamma(a_k))$. Then $\phi_{k+1}^*\gamma_j$ takes the the required form for $1\le j\le k+1$. $\square$

Form this point on we shall treat Q_k and M^k as interchangeable, relating their properties by ϕ_k as necessary. For example, the sections

$$r_k,\ \widetilde{r}_k\colon Q_{k-1}\longrightarrow Q_k$$

are induced by the inclusions of the respective subspaces $(S^3)^{k-1}\times(1,0)$ and $(S^3)^{k-1}\times(0,1)$ of $(S^3)^k$, using the notation of (4.1).

Following [6], we write the facets of I^k as C_h^ε, where $1\le h\le k$ and ε is 0 or 1. Thus C_h^ε is the $(k-1)$–cube $I^{h-1}\times(\varepsilon,1-\varepsilon)\times I^{k-h}$ in $\mathbb{R}^{2k}_{\geqslant}$. Every facet lifts to a codimension–2 submanifold of Q_k, with normal 2–plane bundle ν_h^ε. This is oriented if and only if the corresponding isotropy subcircle $T(C_h^\varepsilon)<T_a^k$ is oriented, since $T(C_h^\varepsilon)$ acts on the normal fibres. An *omniorientation* of Q_k is a choice of orientation for every ν_h^ε; there are therefore 2^{2k} omniorientations in all, and each is preserved by the action of T_a^k.

The Pontryagin-Thom collapse maps $Q_k \to T(\nu_h^\varepsilon)$ determine 2–plane *facial bundles* ρ_h^ε over Q_k. Moreover, an orientation of ν_h^ε determines, and is determined by, an orientation of ρ_h^ε, for every $1 \leq h \leq k$. An omniorientation of Q_k therefore identifies each of the ρ_h^ε as complex line bundles, and reversing any of the constituent orientations induces complex conjugation on the corresponding line bundle.

As explained in [6], there is a canonical isomorphism

$$\tau(Q_k) \oplus \mathbb{R}^{2k} \;\cong\; \bigoplus_{h=1}^{k} \rho_h^0 \oplus \rho_h^1 \tag{4.4}$$

of real $4k$–bundles. Every omniorientation therefore invests the right-hand side with a complex structure, so that (4.4) defines a corresponding stably complex structure on Q_k. These structures play an interesting part in complex cobordism theory, and we shall consider their enumeration in Section 7. As we shall see, they include (3.4).

In [13], Grossberg and Karshon use a noncompact version of (4.2) to describe Bott towers as complex manifolds. Given a list $c = (c_1, \dots, c_{k-1})$ of integral sequences, they construct N_k as the quotient of $(\mathbb{C}^2 \setminus 0)^k$ by a k-fold algebraic torus $\mathbb{C}_\times^k(c)$, under the action

$$\begin{aligned} (w_1, \dots, w_k) \cdot (y_1, z_1; \dots ; y_k, z_k) \;=\; \\ (y_1 w_1, z_1 w_1; y_2 w_2, w_1^{c(1,2)} z_2 w_2; \dots ; y_k w_k, w_1^{c(1,k)} w_2^{c(2,k)} \dots w_{k-1}^{c(k-1,k)} z_k w_k). \end{aligned} \tag{4.5}$$

As complex manifolds, N_k coincides with Q_k, where the latter is determined by the list $a = -c$ (for which $a(i,j) = -c(i,j)$ for all $1 \leq i < j \leq k$). The corresponding structure on M^k is that of the projective form, introduced in Section 3. These observations are used in [9] to relate the quotient cube I^k to the smooth fan determining Q_k.

Note that Grossberg and Karshon's construction yields the bounded flag manifolds B_k when $c_j = (0, \dots, 0, -1)$, where $1 \leq j \leq k-1$.

By mimicing the standard analysis for CP^k [17], we deduce that the corresponding complex tangent bundle $\tau_{\mathbb{C}}(Q_k)$ admits a canonical isomorphism

$$\tau_{\mathbb{C}}(Q_k) \oplus \mathbb{C}^k \;\cong\; (\mathbb{C}^2 \setminus 0)^k \times_{\mathbb{C}_\times^k(a)} \mathbb{C}^{2k}, \tag{4.6}$$

where $\mathbb{C}_\times^k(a)$ acts on $\mathbb{C}^{2k}$ by extending (4.5). The right-hand side splits as the sum of $2k$ complex line bundles, where $\mathbb{C}_\times^k(a)$ acts on $\mathbb{C}$ by

$$y_h \longmapsto y_h w_h \qquad \text{and} \qquad z_h \longmapsto w_1^{-a(1,h)} \dots w_{h-1}^{-a(h-1,h)} z_h w_h,$$

for $1 \leq h \leq k$. Proposition 4.3 identifies these bundles as $\overline{\gamma}_h$ and $\overline{\gamma}_h \otimes \gamma(a_{h-1})$ respectively. So we may rewrite (4.6) as

$$\tau_{\mathbb{C}}(Q_k) \oplus \mathbb{C}^k \;\cong\; \bigoplus_{h=1}^{k} \overline{\gamma}_h \oplus \big(\overline{\gamma}_h \otimes \gamma(a_{h-1})\big). \tag{4.7}$$

The derivation of (4.4) yields isomorphisms $\rho_h^0 \cong \gamma_h$ and $\rho_h^1 \cong \overline{\gamma}_h \otimes \gamma(a_{h-1})$ of real 2–plane bundles. It follows that the stably complex structure (4.7)

arises from an omniorietation of Q_k. The structures induced by the remaining $2^{2k}-1$ omniorientations may then be obtained by replacing approppropriate line bundles by their complex conjugates on the right-hand side of (4.7). We use this procedure to establish (7.2) below.

5. KO-Theory of Stages 1 and 2

The KO-theory of toric manifolds is considerably more subtle than its complex counterpart, and is rarely free over the coefficients. Bahri and Bendersky [2] have obtained interesting results using the Adams Spectral Sequence, although their calculations are mainly additive and make little reference to the geometry of vector bundles. Our goal is to describe $KO^*(M^k)$ as a KO_*-algebra for several families of Bott towers, in terms of the bundles that we have introduced above. We also wish to understand the complexification homomorphism, for application to stably complex structures and cobordism classes in Section 7. Here we focus on M^1 and M^2, which act as base cases for inductive calculation and are useful for establishing notation.

It is convenient to denote the coefficient ring by

$$KO_*=\mathbb{Z}[e,x,y]/(2e,e^3,ex,4x^2-y),$$

where e, x, and y are represented by the real Hopf line bundle over S^1, the symplectic Hopf line bundle over S^4, and the canonical bundle over S^8 respectively [15]. We recall that $KO^*(S^n)$ is a free KO_*-module on the single generator $s_n^{KO}=s_n\in KO^n(S^n)$, such that $s_n^2=0$ for each $n\geq 0$.

We appeal repeatedly to Bott's exact sequence

$$\ldots\longrightarrow KO^{*-1}(X)\xrightarrow{\cdot e}KO^{*-2}(X)\xrightarrow{\chi}K^*(X)\xrightarrow{r}KO^*(X)\longrightarrow\ldots,\tag{5.1}$$

which links real and complex K-theory through the realification homomorphism r. Here, $\cdot e$ denotes multiplication by e, and χ is defined by composing complexification c with multiplication by z^{-1}. For any element g of $K^*(X)$, the difference $g-\overline{g}$ lies in the kernel of r, and hence in the image of χ. Moreover,

$$c(r(g))=g+\overline{g},\quad\text{and}\quad\chi(r(zg))=g-\overline{g}.\tag{5.2}$$

On the other hand, $r(c(h)=2h$ for any h in $KO^*(X)$. It is important to remember that c is multiplicative, whereas r is not.

As in Fujii [11], we define elements u_i in $KO^{-2i}(CP^n)$ by $u_i=r(z^{i+1}u(n))$ for any integer i, where $u(n)\in K^2(CP^n)$ arises in (2.2); as a ring, $KO^*(CP^n)$ may then be described in terms of the u_i. When $n=2$, Fujii's computations stretch to an isomorphism

$$KO^*(CP^2_+)\;\cong\;KO_*[u_i:i\in\mathbb{Z}]/F^2\tag{5.3}$$

of KO_*-algebras, where F^2 is the ideal

$$\big(eu_i,\;xu_i-2u_{i+2},\;u_iu_{2j},\;u_{2i+1}u_{2j-1}-4u_{2(i+j)}:\text{all }i,\,j\big).$$

The relations show that $KO^*(CP^2)$ is free of additive torsion, and that $yu_i=u_{i+4}$ for all i; it therefore suffices to use u_0, u_1, u_2, and u_3, as in [11], but we retain the other u_i for notational convenience. We note that (5.3) actually defines a free K_*-module on a single generator u_i, where zu_i is given by u_{i+1}

for any i. This is equivalent to Wood's well-known result [24] that $KO \wedge CP^2$ is homotopy equivalent to K.

Further computations lead to an isomorphism

$$KO^*(CP^\infty_+) \;\cong\; KO_*[[u_i : i \in \mathbb{Z}]]/F^\infty \tag{5.4}$$

of KO_*-algebras, where F^∞ is the ideal

$$\big(eu_i,\ xu_i - 2u_{i+2},\ u_iu_j - u_{i-2}u_{j+2},\ u_{2i+1}u_{2j-1} - (u_0+4)u_{2(i+j)} : \text{all } i,\ j\big).$$

So $KO^{2n}(CP^\infty)$ is torsion-free, and isomorphic to $u_{-n}\mathbb{Z}[[u_0]]$ for any integer n, whereas $KO^{2n+1}(CP^\infty)$ is zero. For any complex line bundle γ over a 2–generated complex X, it is convenient to interpret the pull-back of u_i along the classifying map of γ as a characteristic class $u_i(\gamma)$ in $KO^{-2i}(X)$.

It follows from (5.4) that $KO_*(CP^\infty)$ is torsion free, and that $KO \wedge CP^\infty$ is homotopy equivalent to the wedge $KO \wedge \big(\bigvee_{k\geq 0} \Sigma^{4k} CP^2\big)$. This equivalence may also be deduced from the fact that a vector bundle is KO-orientable precisely when it is *Spin* [1].

We consider ζ^2 over CP^∞, which is universal for complex line bundles with *Spin*-structure, and utilise the Thom class t^K of Lemma 2.3 in $K^2(T(\zeta^2))$.

Lemma 5.5. *There is a unique element $t_\square$ in $KO^2(T(\zeta^2))$ whose complexification is given by $c(t_\square) = \overline{\zeta}t^K$; it is a Thom class, and satisfies $t_\square^2 = u_{-1}t_\square$ in $KO^4(T(\zeta^2))$.*

Proof. The existence of a Thom isomorphism $KO^{*-2}(CP^\infty_+) \cong KO^*(T(\zeta^2))$ confirms that $KO^{2n}(T(\zeta^2))$ is torsion free for $n \not\equiv 3 \bmod 4$. So (5.1) reduces to a short exact sequence

$$0 \longrightarrow KO^{2n}(T(\zeta^2)) \xrightarrow{\ \chi\ } K^{2n+2}(T(\zeta^2)) \xrightarrow{\ r\ } KO^{2n+2}(T(\zeta^2)) \longrightarrow 0,$$

for $n = 1$ and 2; thus c is monic, and if $t_\square$ exists, it is unique.

The construction of t^K implies that $\overline{t^K} = \overline{\zeta}^2 t^K$, so that

$$c \cdot r(z^{-1}\overline{\zeta}t^K) = z^{-1}(\overline{\zeta}t^K - \zeta\overline{t^K}) = 0;$$

hence $r(z^{-1}\overline{\zeta}t^K) = 0$, and $t_\square$ exists as required. It is a Thom class because $\overline{\zeta}t^K$ is a Thom class and c is a map of ring spectra. Moreover, $(t^K)^2 = z^{-1}(\zeta^2 - 1)t^K$, whence

$$c(t_\square^2) = z^{-1}(1 - \overline{\zeta}^2)t^K = z^{-1}(\zeta - \overline{\zeta})c(t_\square) = c(u_{-1}t_\square).$$

Thus $t_\square^2 = u_{-1}t_\square$ in $KO^4(T(\zeta^2))$. $\square$

The calculation of $KO^*(T(a))$ depends on the parity of a. When $a = 2b$ is even, $\zeta(1)^a$ is *Spin*(2)-bundle, and is the pull-back of the universal example along the map $CP^1 \to CP^\infty$ of degree b; thus $t_\square$ pulls back to a Thom class t in $KO^2(T(a))$.

Proposition 5.6. *When a is even, $KO^*(T(a))$ is isomorphic to*

$$KO_*[s_2, t]/(s_2^2, t^2 - as_2t)$$

as KO_*-algebras. When a is odd, there are elements m_i in $KO^{-2i}(T(a))$ such that $KO^*(T(a))$ is isomorphic to

$$KO_*[m_i : i \in \mathbb{Z}]/F(a, m)$$

as KO_*-algebras, where $F(a,m)$ is the ideal

$$\big(em_i,\ xm_i - 2m_{i+2},\ m_i m_{2j},\ m_{2i+1}m_{2j-1} - 4am_{2(i+j)} : \textit{all } i,\ j\big).$$

Proof. When a is even, the Thom isomorphism identifies $KO^*(T(a))$ with the free KO_*-module on generators t and s_2t. It therefore remains to evaluate t^2 in $KO^4(T(a))$. But $u_{-1}(\zeta(1)^b) = r(z^{-1}(\zeta(1)^b - 1))$ in $KO^2(CP^1)$, so $t^2 = br(z^{-1}(\zeta(1)-1))t = as_2t$, as required.

When $a = 2b+1$ is odd, $\zeta(1)^a$ is no longer KO-orientable. We proceed by comparing the KO-theory of the cofibre sequences of $S^2 \cup_{a\eta} e^4$ and CP^2, using the map $f(a)\colon T(a) \to CP^2$ which classifies $\zeta(1)^a$. We define m_i as $r(z^{i+1}(1 - bzs_2^K)t^K)$ when i is even, and $r(z^{i+1}t^K)$ when i is odd. The action of $f(a)^*$ then yields the algebra structure, by appeal to (5.3); alternatively, we may apply complexification. □

A few observations are in order. Firstly, when a is even the suspension of $a\eta$ is null homotopic, so that $\Sigma T(a)$ is homotopy equivalent to $S^3 \vee S^5$; equivalently, the $SO(3)$-bundle $\mathbb{R} \oplus \zeta(1)^a$ is trivial. Secondly, the relations of Proposition 5.6 imply that $t^3 = 0$. Thirdly, the action of $f(a)^*$ is computed from (5.1), and is given by

$$(5.7)\quad f(a)^*(u_i) = \begin{cases} (2+be^2s_2)t & i=-1 \\ bxs_2t & i=0 \\ xt & i=1 \\ ays_2t & i=2 \end{cases} \quad\text{and}\quad f(a)^*(u_i) = \begin{cases} am_i & i \equiv 0(2) \\ m_i & i \equiv 1(2) \end{cases}$$

for $a = 2b$ and $2b+1$ respectively. Fourthly, when a is odd, the generators m_i may be defined more systematically as $r(z^{i+1}\overline{\zeta}(1)^b t^K)$; this description is central to Theorem 6.7 below.

Proposition 5.6 shows that $KO^*(T(a))$ is free over KO_* when a is even, and over K_* when a is odd. It may be interpreted in terms of spectra as providing homotopy equivalences

$$(5.8)\quad KO \wedge T(2b) \;\simeq\; KO \wedge \big(S^2 \vee S^4\big) \quad\text{and}\quad KO \wedge T(2b+1) \;\simeq\; KO \wedge CP^2.$$

We may now proceed to M^2 via Proposition 3.2, which ensures that there is an additive isomorphism

$$(5.9)\qquad KO^*(M^2) \;\cong\; KO^*(S^2) \oplus KO^*(T(a))$$

of KO_*-modules. It remains to describe the products in $KO^*(M^2)$. To prepare for our eventual notation, we write $p_2^*s_2$ as d_1 in $KO^2(M^2)$ and q_2^*t as d_2 in $KO^2(M^2)$, when a is even; when a is odd, we write $q_2^*m_i$ as n_i in $KO^{-2i}(M^2)$, for all i.

Proposition 5.10. *When a is even, $KO^*(M^2_+)$ is isomorphic to*

$$KO_*[d_1,d_2]\big/(d_1^2,\ d_2^2 - ad_1d_2)$$

as KO_*-algebras; when a is odd, it is isomorphic to

$$KO_*[d_1, n_i : i \in \mathbb{Z}]\,\big/\big(F(a,n),\ d_1^2,\ d_1n_{2i}, d_1n_{2i+1} - 2n_{2i}\big).$$

Proof. It suffices to combine Proposition 5.6 with (5.9). When a is odd, the extra relations follow by applying complexification, and noting that n_i restricts to 0 on M^1 for all i. □

The following corollary is immediate, and helps us to enumerate stably complex structures on M^2 in Section 7.

Corollary 5.11. *In both cases, $KO^{-2}(M^2)$ is isomorphic to $\mathbb{Z}^2$ as abelian groups; bases are given by $\{xd_1, xd_2\}$ when a is even, and $\{xd_1, n_1\}$ when a is odd.*

The isomorphisms of (3.3) extend to $KO^*(M^2)$, and may be described in terms of (5.8) and Proposition 5.10.

6. KO-Theory of Bott Towers

We now return to the Bott tower $(M^k : k \leq n)$, determined by the list $a = (a_1, \ldots, a_{n-1})$, and study inductive procedures for computing the KO_*-algebra structure of $KO^*(M^k)$ in favourable cases.

The work of Bahri and Bendersky [2] identifies the effect of smashing M^k with the spectrum KO, and leads to a homotopy equivalence

$$KO \wedge N^{2n}_+ \;\simeq\; KO \wedge \bigvee_{p,q=0}^{n,n-2} \left(\bigvee^{\alpha_p} S^{2p} \vee \bigvee^{\beta_q} \varSigma^{2q} CP^2 \right) \tag{6.1}$$

for any toric manifold N^{2n}. The *BB-numbers* α_p and β_q enumerate the summands for each p and q respectively. Bahri and Benderskey prove that their numbers are determined by the structure of $H^*(N^{2n};\mathbb{F}_2)$ over $\mathcal{A}(1)$, the subalgebra of the Steenrod algebra generated by Sq^1 and Sq^2. Two types of $\mathcal{A}(1)$-module are involved; the first is $\varSigma^{2p}\mathcal{M}_1$, with one $2p$–dimensional generator on which Sq^1 and Sq^2 act trivially, and the second is $\varSigma^{2q}\mathcal{M}_2$, with one $2q$–dimensional generator x such that $Sq^1x = 0$ and $Sq^2x \neq 0$. Then $H^*(N^{2n};\mathbb{F}_2)$ decomposes as a direct sum of these two types; the number of summands $\varSigma^{2p}\mathcal{M}_1$ is α_p, and the number of summands $\varSigma^{2q}\mathcal{M}_2$ is β_q.

The additive part of our calculations recover (6.1) for two particular families of Bott towers, and provide representative bundles for the generators of $KO^*(M^k)$ as a geometrical bonus. We also point out how the BB-numbers depend on the parity of the entries in a. Our families actually illustrate the extreme cases, which range from $\beta_q = 0$ for all q, to $\alpha_p = 0$ for all $p > 1$.

We begin by reverting to the notation of Section 2, and consider the complex line bundle $\gamma = \gamma_1^{a(1)} \otimes \cdots \otimes \gamma_m^{a(m)}$ over the 2–generated complex X.

When $a(j) = 2b(j)$ is even for all $1 \leq j \leq m$, we write $\gamma_1^{b(1)} \otimes \cdots \otimes \gamma_m^{b(m)}$ as $\gamma^{1/2}$. So γ is $Spin(2)$, and is obtained by pulling the universal example of Lemma 5.5 back along the classifying map for $\gamma^{1/2}$. In particular, we obtain a Thom class $t \in KO^2(T(\gamma))$; it satisfies $t^2 = u_{-1}(\gamma^{1/2})t$, where $u_{-1}(\gamma^{1/2}) = r(z^{-1}(\gamma^{1/2} - 1))$ in $KO^2(X)$, and

$$c(t) = \prod_{j \leq m} (1 + zg_j)^{-b(j)} t^K$$

in $K^0(T(\gamma))$.

Proposition 6.2. *The KO_*-algebra $KO^*(Y_+)$ is a free module over $KO^*(X_+)$ on generators* 1 *and d_{m+1}, which have dimensions* 0 *and* 2 *respectively; the multiplicative structure is determined by the single relation*

$$d_{m+1}^2 = r\big(z^{-1}\big(\prod_{j\leq m}(1+zg_j)^{b(j)}-1\big)\big)d_{m+1}, \tag{6.3}$$

and d_{m+1} restricts to a generator on the fibre $S^2 \subset Y$.

Proof. We repeat the arguments of Lemma 2.3(2) with $q^*t = d_{m+1}$ in $KO^2(Y)$, and apply the remarks above. ☐

It is sometimes preferable to leave (6.3) in the form $d_{m+1}^2 = u_{-1}(\gamma^{1/2})d_{m+1}$, and aim to express $u_{-1}(\gamma^{1/2})$ as a polynomial in the elements $r(z^ig_j)$. This does not follow automatically from (6.3), because r is not multiplicative. The simplest example is $X = S^2$, where $\gamma^{1/2}$ is given by $\zeta^{b(1)}$ and $u_{-1}(\zeta^{b(1)})$ reduces to $2b(1)s_2$ in $KO^2(S^2)$. We then recover the first part of Proposition 5.10.

If one or more of the integers $a(j)$ is odd, the situation is less amenable. For our current purposes, it is enough to recall that $T(\gamma)$ admits a canonical complex line bundle λ over $T(\gamma)$, defined by $v^H(\lambda) = t^H$. So t^K is represented by $z^{-1}(\lambda-1)$ in $K^2(T\gamma)$. The classes $u_i(\lambda)$ in $KO^{-2i}(T(\gamma))$ play a major rôle in describing $KO^*(Y_+)$.

Our main structure theorems refer to two particular families of Bott towers. They are the *totally even* towers, for which the integers $a(i,j) = 2b(i,j)$ are even for all values of $1 \leq i < j \leq n$, and the *terminally odd* towers, for which the integers $a(j-1,j) = 2c(j)+1$ are odd for every $1 \leq j \leq n$. It is possible to deal with other cases by combining the two approaches.

Theorem 6.4. *For any totally even Bott tower $(M^k : k \leq n)$, the KO_*-algebra $KO^*(M_+^k)$ is isomorphic to $KO_*[d_1,\dots,d_k]/J_k^{te}$, where J_k^{te} denotes the ideal*

$$\Big(d_j^2 - r\Big(z^{-1}\Big(\prod_{i<j}(1+zg_i)^{b(i,j)}-1\Big)\Big)d_j : 1 \leq j \leq k\Big);$$

for each $1 \leq k \leq n$, the homotopy equivalence h_k induces the KO_*-module isomorphism

$$KO^*(M^k) \cong \langle d_{\leq 1}\rangle \oplus \cdots \oplus \langle d_{\leq k}\rangle,$$

where $\langle d_{\leq j}\rangle$ denotes the free submodule generated by those monomials d_R for which $R \subseteq [j]$ and $j \in R$.

Proof. In this case the proof of Theorem 3.1 adapts directly, since all the relevant KO_*-modules are free. ☐

As before, it may be preferable to rewrite the relations of J_k^{te} as

$$d_j^2 \;=\; u_{-1}(\gamma_1^{b(1,j)} \otimes \cdots \otimes \gamma_{j-1}^{b(j-1,j)})\, d_j, \tag{6.5}$$

and calculate $u_{-1}(\gamma_1^{b(1,j)} \otimes \cdots \otimes \gamma_{j-1}^{b(j-1,j)})$ as a polynomial in $d_1, \dots, d_{j-1}$ for each $1 \leq j \leq k$. Amongst other formulae in $KO^*(M^k)$, this approach yields

$$d_j^{j+1} = 0 \quad \text{and} \quad d_j^2 = \big(a(1,j)d_1 + \cdots + a(j-1,j)d_{j-1}\big)d_j \;\text{ modulo } P_*,$$

where P_* denotes the ideal generated by triple products.

Calculations for terminally odd towers are more intricate, and we begin with the additive structure. It is convenient to index the generators by finite sets R of positive integers. For every such R, we construct R^+ by adding 1 to each element, and $1;R^+$ by adjoining the integer 1 to the result. We obtain the coproduct decomposition

$$2^{[k-1]} \xrightarrow{e_1} 2^{[k]} \xleftarrow{e_2} 2^{[k-1]}, \tag{6.6}$$

of power sets, where $e_1(R) = R^+$ and $e_2(R) = 1;R^+$. Given $R \subseteq [k-2]$ for $k \geq 2$, we construct $R;k \subset [k]$ by adjoining the integer k.

So far as complex K-theory is concerned, we may apply this notation to the ladder (3.7). The elements $g_R t_k^K$ in $K^*(T(a_{k-1}))$ are of two types; those for which R takes the form S^+ for some $S \subseteq [k-3]$, so that $i^*(g_R t_k^K) = g'_S(t')_{k-1}^K$, and those for which R takes the form $1;S^+$, so that $f^*(\Sigma^2 g'_S(t')_{k-1}^K) = g_R t_k^K$. The decomposition (6.6) then corresponds to the splitting (3.8). Of course, $q_k^*(g_R t_k^K) = g_{R;k}$ in $K^*(M^k)$.

We may now construct the elements we need in KO-theory. For every integer i, we define

$$m(R;k)_i \;=\; r\big(z^{i+1}\overline{\gamma}_{k-1}^{\,b(k)}\, g_R t_k^K\big)$$

in $KO^{2(|R|-i)}(T(a_{k-1}))$, as R ranges over subsets of $[k-2]$, and

$$n(R;j)_i \;=\; r\big(z^{i+1}\overline{\gamma}_{j-1}^{\,b(j)}\, g_{R;j}\big)$$

in $KO^{2(|R|-i)}(M^k)$, as R ranges over subsets of $[j-2]$, with $2 \leq j \leq k$. Thus $q_k^* m(R;k)_i = n(R;k)_i$ for every $R \subseteq [k-2]$.

Theorem 6.7. *For any terminally odd Bott tower $(M^k : k \leq n)$, the KO_*-module $KO^*(M_+^k)$ is generated by the elements*

$$\big\{d_1,\ n(R;j)_i : 2 \leq j \leq k\big\},$$

where R ranges over the subsets of $[j-2]$ and $i \in \mathbb{Z}$; the submodule of relations is generated by

$$\big\{en(R;j)_i,\ xn(R;j)_i - 2n(R;j)_{i+2}\big\}$$

for all R, j and i.

Proof. We proceed inductively, using the commutative ladder (3.7). We assume that the result holds for terminally odd towers of height $\leq n-1$, where $n \geq 2$, and consider $(M^k : k \leq n)$, determined by a list $(a_1, \ldots, a_{n-1})$. The tower $((M')^k : k \leq n-1)$ is determined by the list $(a'_1, \ldots, a'_{n-2})$, where a'_{j-1} is obtained from a_j by deleting the first element; so it is also terminally odd, and the inductive hypothesis applies.

We may therefore assume that $KO^*(T(a'_{k-2}))$ is a free abelian group, generated by the elements $m(S;k-1)'_i$ for $S \subseteq [k-3]$ and $0 \leq i \leq 3$. So $KO^*(\Sigma^2 T(a'_{k-2}))$ is generated by their double suspensions, and both groups are zero in odd dimensions. Since $i^*\gamma_j = \gamma'_{j-1}$ in $KO^0((M')^{k-2}$ for every $2 \leq j \leq k-1$, it follows from Corollary 3.6 that $i^* m(S^+;k)_i = m(S;k-1)'_i$ in

$KO^0((M')^{k-2}$, and $f^*(\Sigma^2 m(S;k-1)'_i = m(1;S^+;k)_i$ in $KO^*(T(a_k))$, for every $S \subseteq [k-3]$. Applying $KO^*(-)$ to the ladder yields

$$\begin{tikzcd} {} & KO^*(T(a'_{k-2})) \arrow[l, "\delta"'] \arrow[d, "(q'_{k-1})^*"'] & KO^*(T(a_{k-1})) \arrow[l, "i^*"'] \arrow[d, "q_k^*"'] & KO^*(\Sigma^2 T(a'_{k-2})) \arrow[l, "f^*"'] \arrow[d, "\Sigma^2(q'_{k-1})^*"] & {} \arrow[l, "\delta"'] \\ {} & KO^*((M')^{k-1}) \arrow[l, "\delta"] & KO^*(M^k) \arrow[l, "i^*"] & KO^*(\Sigma^2(M')^{k-1}_+) \arrow[l, "f^*"] & {} \arrow[l, "\delta"] \end{tikzcd},$$

ensuring that the upper coboundary maps δ are zero for $k \geq 2$, and that the upper sequence splits as abelian groups. So $KO^*(T(a_{k-1}))$ is also zero in odd dimensions, and generated by the $m(S^+;k)_i$ and $m(1;S^+;k)_i$ in even dimensions; but these are precisely the elements $m(R;k)_i$ for $R \subseteq [k-2]$. It follows from Proposition 3.2 that q_k^* injects $KO^*(T(a_{k-1}))$ into $KO^*(M^k)$ as the summand generated by the elements $n(R;k)_i$, for $R \subseteq [k-2]$. The abelian group structure of $KO^*(T(a_{k-1}))$ ensures that complexification is monic, and therefore that

$$en(R;k)_i = 0 \quad \text{and} \quad xn(R;k)_i = 2n(R;k)_{i+2}$$

for all $2 \leq j \leq k$. The remainder of the additive structure then follows from the inductive hypothesis. The base case $k = 2$ is resolved by Proposition 5.10, with $m(\varnothing;2)_i = m_i$ and $n(\varnothing;2)_i = n_i$ for all i. □

It follows from Theorem 6.7 that $KO^*(M^k)$ is torsion free, except for a single copy of $\mathbb{Z}/2$ in each of the dimensions 0 and 1 mod 8, generated by $e^2y^id_1$ and ey^id_1 respectively. This generalises the results obtained for B_k in [9], and outlined in Example 6.13 below. Theorem 6.7 also implies that $yn(R;j)_i = n(R;j)_{i+4}$ in $KO^*(M^k)$, for any R, j, and i. We may therefore restrict the choice of generators to $i = 0$, 1 2, and 3, for example; nevertheless, we usually allow i to be arbitrary for notational convenience.

In order to understand the multiplicative structure of $KO^*(M^k)$, we need to evaluate products of the generators described in Theorem 6.7.

Proposition 6.8. *For any $R \subseteq [j-2]$ and $j > 2$, we have that*

$$d_1 n(R;j)_i = \begin{cases} 0 & \text{if } 1 \in R \\ n(1;R;j)_i & \text{otherwise} \end{cases};$$

for any $R' \subseteq [j'-2]$, we have that

$$\begin{aligned} n(R;j)_i\, n(R';j')_{i'} &= \\ & r\Big(z^{i+j+2}\overline{\gamma}_{j-1}^{b(j)} g_{R;j}\Big(\overline{\gamma}_{j'-1}^{b(j')} g_{R';j'} + (-1)^{j+1}\gamma_{j'-1}^{b(j')}\overline{g}_{R';j'}\Big)\Big). \end{aligned} \tag{6.9}$$

In particular, $n(R;j)_i\, n(R';j)_{i'} = 0$ whenever $1 \in R \cap R'$.

Proof. Theorem 6.7 implies that complexification is monic, modulo the summand $KO^*(M^1)$. Since $n(R;j)_i$ restricts to 0 in $KO^*(M^1)$ for every R, j, and i, it suffices to prove the relations by applying c.

Now $c(d_i) = g_1$, and $c(n(R;j)_i) = z^{i+1}(\overline{\gamma}_{j-1}^{b(j)} g_{R;j} + (-1)^{i+1}\gamma_{j-1}^{b(j)}\overline{g}_{R;j})$ in $K^*(M^k)$. Moreover, $g_1^2 = 0$, so $\overline{g}_1 = g_1$ and the first set of relations follows.

The second set is proven similarly, by noting that

$$c\big(r(x)r(y)\big) = cr(x(y+\overline{y}))$$

for any elements x and y in $KO^*(M^k)$. $\square$

We would like to write (6.9) as an explicit KO_*-linear combination of the generators d_1 and $n(R;j)_i$. In principle, this may be achieved by using the expressions for $\overline{g}_m$ and g_m^2 of (2.14) and (2.15) respectively; in practice, the calculations increase rapidly in complexity. Examples 6.11 and 6.13 give a more detailed glimpse of the difficulties which characterise the multiplicative structures described in Theorem 6.4 and Proposition 6.8. Related calculations will be presented in [10].

The following observations flow directly from Theorems 6.4 and 6.7.

Corollary 6.10. *In the totally even case, the equivalence* (6.1) *reduces to*

$$KO \wedge M_+^k \;\simeq\; KO \wedge \bigvee_{R\subseteq [k]} S^{2|R|}\,;$$

thus $\alpha_p = \binom{k}{p}$ for all $1 \le p \le k$, and $\beta_q = 0$ for all q. In the terminally odd case, we have

$$KO \wedge M_+^k \;\simeq\; KO \wedge \left(S_+^2 \vee \bigvee_{h=0}^{k-2} \bigvee_{R\subseteq[h]} \Sigma^{2|R|}\, CP^2 \right);$$

thus $\alpha_p = 0$ for $2 \le p \le k$, and $\beta_q = \sum_{h=q}^{k-2} \binom{h}{q}$ for all $0 \le q \le k-2$.

Proof. In the totally even case, Theorem 6.4 confirms that $KO^*(M_+^k)$ is additively generated over KO_* by the monomials $d_R = \prod_R g_j$, as R ranges over the subsets of $[k]$.

In the terminally odd case, the torsion subgroup of $KO^*(M^k)$ corresponds to the summand $KO \wedge S^2$. The proof of Theorem 6.7 combines with (5.8) to show that

$$KO \wedge T(a_{j-1}) \;\simeq\; KO \wedge \Big((S_+^2)^{\wedge (j-2)} \wedge CP^2 \Big)$$

for all $1 \le j \le k$, where the elements $n(R;j)_i$ correspond to the summand $\Sigma^{2|R|} CP^2$ for every $R \subseteq [j-2]$. The result now follows from Proposition 3.2. $\square$

Corollary 6.10 illustrates the relationship between the BB-numbers and entries in the list a. In the totally even case, Proposition 3.1 confirms that every square is zero in $H^*(M^k;\mathbb{F}_2)$, so $Sq^2 = 0$; thus $\Sigma^{2q}\mathcal{M}_2$ cannot occur in its decomposition, and $\beta_q = 0$ for all $1 \le j \le k$, as required. In the terminally odd case, we write the mod 2 reduction of the class x_i as x_i'. Then Proposition 3.1 confirms that $Sq^2 x_1' = 0$, and $Sq^2 x_j' \equiv x_{j-1}' x_j'$ modulo terms of the form $x_i' x_j'$ with $i \le j-2$, for every $2 \le j \le k$. Thus $\alpha_1 = 1$. A simple inductive calculation reveals that $H^{2q+2}(M^k;\mathbb{F}_2)$ decomposes as

$$Sq^2 H^{2q}(M^k;\mathbb{F}_2) \oplus H_{2q+2},$$

where H_{2q+2} is generated by all monomials of the form $x'_R x'_j$ such that $R \subseteq [j-2]$ and $|R| = q$. Since Sq^2 is injective on H_{2q+2}, it follows that $\alpha_p = 0$ for $2 \leq p \leq k$, and $\beta_q = \sum_{h=q}^{k-2} \binom{h}{q}$ for all q, as required.

In order to illustrate these results, we discuss two examples.

Example 6.11. Let $(A_k : k \geq 0)$ be the totally even tower determined by the integers $a(i,j) = 0$ for $i \leq j-2$, and $a(j-1,j) = 2$, for any $j \geq 1$. The relation (6.5) reduces to $d_j^2 = u_{-1}(\gamma_{j-1})d_j$, so we have to compute $u_{-1}(\gamma_{j-1})$ in $KO^2(M^j)$; this follows inductively from an understanding of the homomorphism $f^*\colon KO^*(CP^\infty) \to KO^*(T(\zeta^2))$, where f is the map of Thom complexes classifying ζ^2. To calculate f^*, we extend the formulae of (5.7) in case $b = 1$, and find

$$f^*(u_i) = \begin{cases} (2+u_0)t_\square & i = -1 \\ (e^2+u_1)t_\square & i = 0 \\ (x+u_2)t_\square & i = 1 \\ u_3 t_\square & i = 2 \end{cases}. \tag{6.12}$$

We deduce that $u_{-1}(\gamma_{j-1})$ is given by

$$\sum_{s=0}^{\lfloor\frac{j-1}{4}\rfloor} 2y^s d_{j-1}\ldots d_{j-4s} + \sum_{s=0}^{\lfloor\frac{j-2}{4}\rfloor} e^2 y^s d_{j-1}\ldots d_{j-4s-1} + \sum_{s=0}^{\lfloor\frac{j-4}{4}\rfloor} xy^s d_{j-1}\ldots d_{j-4s-3}.$$

Example 6.13. Let $(B_k : k \geq 0)$ denote the terminally odd tower of bounded flag manifolds, determined by integers $a(i,j) = 0$ for $i \leq j-2$ and $a(j-1,j) = 1$, for all $j \geq 1$. Then each $b(j)$ is zero, and the generators $n(R;j)_i$ are defined by $r(z^{i+1}g_{R;j})$ for every $R \subseteq [j-2]$. Products of the form $n(R;j)_i \cdot n(R';j')_{i'}$ are given by

$$r\big(z^{i+j+2}(g_{R;j}\, g_{R';j'} + (-1)^{j+1} g_{R;j}\, \overline{g}_{R';j'})\big),$$

and are evaluated using the formulae

$$g_m^2 = \left(\sum_{\varnothing \neq S \subseteq [m-1]} z^{|S|-1} g_S\right) g_m \qquad \text{and} \qquad \overline{g}_m = g_m/(1+zg_m)$$

in $K^*(M^k)$, for every $1 \leq m \leq k$.

We may combine Theorems 6.4 and 6.7 to identify $KO^{-2}(M^k)$. As explained in Section 7, these groups classify the stably almost complex structures on M^k.

Theorem 6.14. *If the tower is totally even, then $KO^{-2}(M^k)$ is isomorphic to*

$$\Big(\bigoplus_{|R|\equiv 1,-1(4)} \mathbb{Z}\Big) \oplus \Big(\bigoplus_{|R|\equiv 0(4)} \mathbb{Z}/2\Big),$$

where $R \subseteq [k]$; a basis is given by

$$\big\{xy^{(|R|-1)/4}d_R,\ y^{(|R|+1)/4}d_R,\ e^2y^{|R|/4}d_R\big\}.$$

If the tower is terminally odd, then $KO^{-2}(M^k)$ is isomorphic to $\mathbb{Z}^{2^{k-1}}$; a basis is given by $\big\{xd_1, n(R;j)_i\big\}$, where $R \subseteq [j-2]$ for $2 \leq j \leq k$ and $i = |R|+1$.

7. Stably Complex Structures

By way of conclusion, we apply our results to the study of stably complex structures on certain families of Bott towers. We consider the enumeration of those which arise from omniorientations, and discuss two particular special cases; those which restrict to almost complex structures, and those which are null-cobordant in Ω^U_*. We summarise the appropriate definitions in order to establish our notation.

Full details of the results for $(B_k : k \geq 0)$ in Theorems 7.3, 7.6 and 7.8 are provided in [8].

We write BU and BO respectively for the classifying spaces of the infinite unitary and orthogonal groups, and let $r\colon BU \to BSO \subset BO$ denote a specific choice of realification. The resulting maps

$$SO/U \xrightarrow{f} BU \xrightarrow{r} BO$$

induce the K-theory exact sequence (5.1) for connected spaces X. Given a smooth oriented manifold N, we assume that the stable tangent bundle is represented by a map $\tau^S\colon N \to BSO$, which we fix henceforth. A complex structure on τ^S is given by a lift τ to BU, and is known as a stably complex structure, or *U-structure*, on N; it therefore consists of a factorisation $\tau^S = r \cdot \tau$. We deem two U-structures τ and τ' to be equivalent, or *homotopic*, whenever they are homotopic through lifts of τ^S. Once τ is chosen, it leads to a complementary lift of the stable normal bundle ν^S of N, and conversely; this correspondence preserves homotopy classes.

If we begin with the opposite orientation for N, we obtain a second set of U-structures and homotopy classes. They are distinct from those described above, but correspond to them bijectively.

An *almost complex* structure on N is given by a complex structure on the tangent bundle $\tau(N)$, and determines a compatible orientation. When N is a complex manifold, it therefore admits a corresponding almost complex structure, which stabilises to the *underlying* U-structure $\tau_{\mathbb{C}}$. An arbitrary U-structure need not, of course, destabilise to $\tau(N)$, just as an almost complex structure need not be integrable. Henceforth, we will deal only with complex connected N, oriented compatibly, and will take $\tau_{\mathbb{C}}$ to be the distinguished U-structure. As explained in [19], we may then define a bijection between $KO^{-2}(N)$ and the homotopy classes of U-structures on N. To each $\Delta \in KO^{-2}(N)$ there corresponds a homotopy class of complex structures on the trivial bundle $\mathbb{R}^{2L}$, for suitably large L, and the bijection associates the U-structure $\tau := \tau_{\mathbb{C}} \oplus \mathbb{R}^{2L}$ to Δ. In other words, $\Delta(\tau, \tau_{\mathbb{C}})$ is the *difference element* of τ; its image under χ is represented by the virtual bundle $\tau - \tau_{\mathbb{C}}$ in $K^0(N)$.

So Theorem 6.14 identifies the totality of U-structures on the Bott tower $(M^k : k \leq n)$. In the terminally odd case, χ is monomorphic and the structures may be enumerated by identifying $\tau_{\mathbb{C}}$ as an element of $K^0(M^k)$, then varying $\tau - \tau_{\mathbb{C}}$ over the image of χ. This strategy was applied to the tower of bounded flag manifolds $(B_k : 0 \leq k)$ in [9].

For more general purposes, it helps to follow the lead of Section 4, and define a complex structure on an arbitrary vector bundle θ as an isomorphism g from

θ to a complex vector bundle ξ. The action of i on the fibres of θ is given by conjugating its action on ξ by g, and homotopy classes of isomorphisms correspond to homotopy classes of complex structures. An isomorphism of the form $\tau(N) \oplus \mathbb{R}^m \cong \xi$ therefore specifies a U-structure on N; for example, (4.7) defines the U-structure $\tau_{\mathbb{C}}$ underlying the projective form of M^k.

A second isomorphism $g' \colon \theta \cong \xi$ defines a second complex structure θ', which differs stably from the first by a unique difference element $\Delta(\theta', \theta)$ in $KO^{-2}(N)$. As above, its image under χ is represented by the virtual bundle $\theta' - \theta$ in $K^0(N)$. Whether or not χ is monic, $\Delta(\theta', \theta)$ is constructed by expressing the trivial bundle $\mathbb{R}^{2L}$ as $\theta \oplus \theta^{\perp}$ for suitably large L, then taking the complex structure induced by g' on θ and by the Hermitian complement of g on $\theta^{\perp}$. We are particularly interested in this situation when g' is obtained from g by complex conjugation; the difference element may then be described as follows.

Lemma 7.1. *For any complex vector bundle ξ over N, the difference element $\Delta(\overline{\xi}, \xi)$ is given by $r(z(\overline{\xi} - 1))$ in $KO^{-2}(N)$.*

Proof. It is sufficient to consider the universal bundle υ over a complex Grassmannian of the form $U(W \oplus W')/U(W) \times U(W')$, where $W \oplus W'$ is isomorphic to $\mathbb{C}^L$ for suitably large L. Both $\Delta(\overline{\upsilon}, \upsilon)$ and $r(z(\overline{\upsilon} - \mathbb{C}))$ may be represented by maps into $\Omega^2 SO(W \oplus W')$, obtained by adjointing Bott's original periodicity maps. Details of these are in [7], as are the techniques for proving that the two maps are homotopic. $\square$

For any Bott tower $(M^k : k \leq n)$, we write $o(a, k)$ (or $o(k)$ when the list a is understood or irrelevant) for the number of homotopy classes of U-structures which arise from the omniorientations of M^k. Thus $1 \leq o(k) \leq 2^{2k}$. Applying Lemma 7.1 and the splitting (2.8) to the U-structure $\tau_{\mathbb{C}}$ of (4.7) identifies the corresponding difference elements as

$$\begin{aligned} &\sum_{j=1}^{k} \delta_j \Delta(\gamma_j, \overline{\gamma}_j) + \sum_{j=1}^{k} \epsilon_j \Delta\big((\overline{\gamma}(a_{j-1}) - \overline{\gamma}_j), (\gamma(a_{j-1}) - \gamma_j)\big) \ = \\ &\qquad \sum_{j=1}^{k} (\delta_j + \epsilon_j) r(z^2 g_j) - \sum_{j=1}^{k} \epsilon_j r\Big(z^2 \prod_{i<j} (g_i + 1)^{a(i,j)}\Big), \end{aligned} \tag{7.2}$$

where δ_j and ϵ_j are 0 or 1 for all $1 \leq j \leq k$.

When $k = 1$, these reduce to 0, xd_1 and $2xd_1$ in $KO^{-2}(M^1)$, so that $o(1) = 3$. When $k = 2$, Corollary 5.11 shows that we obtain the same elements, together with their translates by

$$xd_2, \quad x(d_2 - a(1,2)d_1), \quad \text{and} \quad x(2d_2 - a(1,2)d_1)$$

when $a(1,2)$ is even, and

$$n_{2,1}, \quad n_{2,1} - a(1,2)xd_1, \quad \text{and} \quad 2n_{2,1} - a(1,2)xd_1$$

when $a(1,2)$ is is odd. So $o(a, 2) = 9$, 10, 11, and 12, as $a(1,2) = 0$, ± 1, ± 2, and $|a(1,2)| \geq 3$ respectively.

The calculations increase rapidly in complexity for general values of $a(i,j)$. Nevertheless, certain families of special cases yield interesting conclusions.

Theorem 7.3. *For any Bott tower $(M^k : k \leq n)$, we have that*

$$3^k \leq o(k) \leq 3 \cdot 4^{k-1}$$

for each $1 \leq k \leq n$. The maximum is attained by any tower for which the inequality $|a(k-1,k)| \geq 3$ holds for all k, and the minimum by the tower $((CP^1)^k : k \geq 0)$; the tower of bounded flag manifolds $(B_k : k \geq 0)$ satisfies

$$o(k) = \sum_{i=0}^{\lceil k/2 \rceil} \binom{k+1}{2i} 2^{k-i}.$$

Proof. We proceed by induction on k, having resolved the cases $k = 1$ and 2 above. We assume first that $|a(k-1,k)| \geq 3$ for all k, and that $o(k-1) = 3 \cdot 4^{k-2}$. For M^k, the difference elements (7.2) consist of pullbacks from M^{k-1}, plus their translates by the three nonzero elements

$$(\delta_k + \epsilon_k) r(z^2 g_k) \; + \; \delta_k r\Big(z^2 \prod_{j<k} (g_j + 1)^{a(j,k)}\Big). \tag{7.4}$$

These map to $(\delta_k + \epsilon_k)(g_k - \overline{g}_k) + \delta_k \big(\prod_{j<k}(g_j+1)^{a(j,k)} - \prod_{j<k}(\overline{g}_j+1)^{a(j,k)}\big)$ under complexification, where $-\delta_k a(k-1,k)(g_{k-1} - \overline{g}_{k-1})$ is the only term involving g_{k-1}. It follows that no such translates can result in coincident difference elements when $|a(k-1,k)| \geq 3$, and the initial induction is complete.

The tower $((CP^1)^k : k \geq 0)$, on the other hand, has $a(i,j) = 0$ for all values of i and j, and is totally even. The translation elements (7.4) then reduce to $(\delta_k + \epsilon_k) x d_k$, creating one coincidence for each element pulled back from $(CP^1)^{k-1}$; this maximises the possible coincidences, and leads to $o(k) = 3o(k-1)$. So $o(k) = 3^k$, represented by the difference elements $\sum_{j=1}^{k} \omega_j x d_j$, where $\omega_j = 0$, 1, or 2 for each j.

The tower $(B_k : k \geq 0)$ has $a(j-1,j) = 1$ for all $j < k$, and $a(i,j) = 0$ otherwise. Being terminally odd, we may follow Theorem 6.14, and work with the complexifications

$$g_k - \overline{g}_k, \; -(g_{k-1} - \overline{g}_{k-1}) + (g_k - \overline{g}_k), \text{ and } -(g_{k-1} - \overline{g}_{k-1}) + 2(g_k - \overline{g}_k),$$

of the translation elements (7.4). These yield two coincidences for each element of the $(k-2)$th stage. In other words, $o(k)$ satisfies the difference equation $o(k) = 4o(k-1) - 2o(k-2)$ for each $k \geq 2$. Using the initial conditions provided by $k = 1$ and 2, we may then apply standard techniques [16] to deduce the required formula. The same arguments work when $a(j-1,j) = -1$ and $a(i,j) = 0$ for $i \neq j-1$. □

We emphasise that these results depend on our initial choice of orientation for M^k, as do Theorems 7.6 and 7.8 below.

It transpires that the U-structure τ' of (3.4) is amongst those induced by an omniorientation, whose difference element satisfies $\delta_j = 1$ and $\epsilon_j = 0$ in (7.2), for all $1 \leq j \leq k$.

Theorem 7.5. *For any Bott tower $(M^k : k \leq n)$, the difference element $\Delta(\tau', \tau_{\mathbb{C}})$ is given by $\sum_{j=1}^{k} r(z^2 g_j)$ in $KO^{-2}(M^k)$.*

Proof. We proceed by induction on k, choosing $k = 0$ as the base case because the elements in question are both zero.

So we assume that the result is true For M^{k-1}, and consider the construction of M^k. We observe that τ' and $\tau_{\mathbb{C}}$ both arise by pulling back the corresponding U-structures on M^{k-1}, and adding the bundle of tangents along the fibres. By induction, the structures on M^{k-1} differ by $\sum_{j=1}^{k-1} r(z^2 g_j)$. Moreover, the tangents along the fibres pull back from the corresponding bundles along the fibres of the universal example over CP^∞. In this case, $KO^*(CP^\infty)$ is torsion free, so that χ is monic and we may work in $K^0(CP^\infty)$. The relevant difference element is therefore $r(z^2 u)$, and pulls back to $r(z^2 g_k)$ over M^k. Adding the results yields the required formula. □

The structure $\tau_{\mathbb{C}}$ is the stabilisation of an almost complex structure, and we would like to estimate how many others that are induced by an omniorientation share this property. We recall from Section 3 our observation that the Euler characteristic $e(M^k)$ is 2^k.

According to Thomas [22], the structures we seek are precisely those whose kth Chern class coincides with $e(M^k)$, and therefore with $c_k(\tau_{\mathbb{C}})$. We may compute the latter by combining (2.8) with (4.7) and writing the total Chern class $c(\tau_{\mathbb{C}})$ as

$$(1 - 2x_1)\prod_{j=2}^{k}(1 + a(1,j)x_1 + \cdots + a(j-1,j)x_{j-1} - 2x_j).$$

We deduce that $c_k(\tau_{\mathbb{C}}) = (-2)^k x_1 \ldots x_k$. This confirms the value of $e(M^k)$, and shows that the orientation class defined by the complex structure on the projective form of M^k is the dual of $(-1)^k x_1 \ldots x_k$ in $H_{2k}(M^k : \mathbb{Z})$.

Theorem 7.6. *For any Bott tower $(M^k : k \leq n)$, the omniorientations induce 2^{k-1} distinct almost complex structures on M^k, for each $1 \leq k \leq n$.*

Proof. We may build up the total Chern class of every U-structure on M^k by analogy with the proof of Theorem 7.3; when $k = 1$ we obtain $1 - 2x_1$, $1 + 2x_1$ or 1. Only the first of these has the required c_1, confirming the result for $k = 1$.

To obtain the kth stage, we multiply the $(k-1)$th stage by one of the four possible factors

$$\begin{aligned} &1 \pm \big(a(1,k)x_1 + \ldots + a(k-1,k)x_{k-1}\big) \quad \text{or} \\ &\qquad\qquad 1 \pm \big(a(1,k)x_1 + \cdots + a(k-1,k)x_{k-1} - 2x_k\big). \end{aligned} \tag{7.7}$$

The only way in which the monomial $x_1 \ldots x_k$ (or any of its equivalent forms such as x_k^k) can occur in the final product is by selecting one of the latter two factors at this, and every previous, stage. There are 2^k such possibilities in all, distributed equally between $\pm 2^k x_1 \ldots x_k$.

It remains only to prove that there are no repetitions amongst the 2^{k-1} products with sign $(-1)^k$. In fact all 2^k structures have distinct c_1, as a simple computation shows. □

The relevance of bounded flag manifolds to complex cobordism theory was first highlighted in [18]. Somewhat surprisingly, the most important U-structure

from this point of view is τ', which bounds. We would therefore like to know how many bounding U-structures arise from the omniorientions of M^k. We denote this number by $b(k)$, and conclude with a brief analysis of its possible values.

Theorem 7.8. *For any Bott tower $(M^k : k \leq n)$, we have that*

$$3^{k-1} \leq b(k) \leq 3 \cdot 4^{k-1} - 2^k$$

for each $1 \leq k \leq n$. The towers $((CP^1)^k : k \geq 0)$ and $(B_k : k \geq 1)$ satisfy

$$b(k) = 3^k - 2^k \quad \text{and} \quad b(k) = \sum_{i=0}^{\lceil k/2 \rceil} \binom{k}{2i-1} 2^{k-i}$$

respectively.

Proof. The lower bound arises from Theorem 7.3 by applying Szczarba's construction [21] to deduce that every U-structure on M^{k-1} lifts to a bounding U-structure on M^k. The upper bound arises from the fact that the kth Chern number $c_k[M^k]$ of every bounding U-structure is zero. Applying (7.7) shows that $c_k[M^k] \neq 0$ for precisely 2^k distinct U-structures, and the inequality $b(k) \leq 3 \cdot 4^{k-1} - 2^k$ then follows from Theorem 7.3.

The 3^k distinct U-structures on $(CP^1)^k$ arise by choosing one of the three possible structures for each factor CP^1; one bounds, the other two do not. A structure on the product bounds precisely when one or more of these k choices bound, yielding $b(k) = 3^k - 2^k$. For B_k, we note from the proof of Theorem 7.3 that

$$b(k) = 2o(k-1) - 2o(k-2),$$

so $b(k)$ satisfies $b(k) = 4b(k-1) - 2b(k-2)$. But there are no bounding U-structures on a point, and only one on M^1; so $b(0) = 0$ and $b(1) = 1$. Solving the difference equation gives the required formula. □

Many interesting questions remain to be answered about the rôle of Bott towers in complex cobordism theory. We hope to return to these in future.

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